



Extended T-Duality In Three Dimensions

Dualidad T Extendida En Tres Dimensiones

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Resumen

Se extiende el concepto de *Dualidad de Wick* de la Relatividad general en $2 + 1$ dimensiones (con constante cosmológica) a la acción efectiva de bajas energías de la teoría de cuerda bosónica en $2 + 1$ dimensiones. Llamamos a esta extensión “Dualidad de Wick para cuerdas” y probamos que es una simetría de la acción efectiva de bajas energías. La estructura de esta ‘dualidad’ es estudiada al componerla con la dualidad de Buscher a lo largo de un vector de Killing de tipo tiempo en una solución tipo agujero negro BTZ sin rotación, probándose que ambas dualidades conmutan. Dada la naturaleza topológica de la teoría de cuerdas en tres dimensiones, argumento que la dualidad de Wick para cuerdas constituye una simetría exacta de la teoría completa.

Palabras Clave: Agujero negro; Dualidad; Teoría de supercuerdas.

Abstract

The concept of ‘Wick duality’ of General Relativity in $2 + 1$ dimensions with a cosmological constant is extended to the low energy effective action of bosonic string theory in $2 + 1$ dimensions. We call this extension ‘stringly Wick duality’ and we show that it is a symmetry of the low energy effective action. The structure of this ‘duality’ is studied by composing it with the Buscher duality along a time-like killing vector of the nonrotating BTZ Black Hole solution, and it is proved that the action of both dualities commutes. Given the topological nature of string theory in three dimensions, it is argued that the ‘stringly Wick duality’ must be an exact symmetry of the entire theory.

Keywords: Black hole; Duality; Superstring Theory.

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1 Introduction

Duality is a very important tool in the study of nonperturbative physics in quantum field and string theories (for a review, see for instance [1]). For example, duality helps to describe the strong coupling limit of some supersymmetric field and string theories. Thus, it is important to determine if a theory does admit dual versions. In order to do that, the Roček-Verlinde procedure can be applied for every field theory with a *global* symmetry [9]. This global symmetry

can be made local to construct a more general Lagrangian with additional variables (Lagrange multiplier fields) and a bigger symmetry. From this *parent* Lagrangian, the original Lagrangian and its associated dual Lagrangian can be obtained. This global symmetry can be abelian or non-abelian and, according to it, the above mentioned duality algorithm is called, abelian or non-abelian duality respectively (for a review see [2]).

On the other hand, Chern-Simons gauge theory has been used to describe $(2 + 1)$ -dimensional gravity [3]. On the

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mathematical side, Chern-Simons theory has been very useful for constructing knots and links invariants [4]. This theory possesses a duality which works by inverting the level k to $\frac{1}{k}$. In Ref. [5], this was studied for the abelian case. The generalization to the non-abelian case has been worked out in Ref. [6] in the context of gravitational duality.

In generic terms, the physical principles behind string theory remain still unknown. One way to look for them is through the searching of new symmetry principles of the theory. Thus it is very important to explore new symmetries or extend some of the already known string theory symmetries: like Buscher's duality or T-duality. In order to accomplish that, critical (super)string theory is rather complicated and toy models would be more useful. One of these models is the topological non-linear sigma model.

Here we take a basic toy model given by string gravity in $2 + 1$ dimensions and will explore the extension of T-duality by considering topological symmetries of its corresponding Chern-Simons representation. This topological symmetry is the Wick duality and we show that it is compatible with T-duality.

2 T Duality

An example of the algorithm of duality described before is the so called "Buscher Duality" [8, 9]. This duality appears when the algorithm is applied to the standard bosonic string action. In this way it is convenient to remember that the Polyakov action can be generalized to be consistent with all the symmetries and the massless spectrum of the closed bosonic strings in the form of a non linear sigma model

$$\hat{S} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{h} \left[\left(h^{ab} G_{IJ}(X) + i\varepsilon^{ab} B_{IJ}(X) \right) \times \partial_a X^I \partial_b X^J + \alpha' \Phi(X) R^{(2)} \right] \quad (1)$$

where $G_{IJ}(X)$ is an arbitrary metric of the background space, $B_{IJ}(X)$ is the antisymmetric tensor and $\Phi(X)$ is the dilaton. The integration is performed on the two dimensional surface Σ and ε^{ab} is the volume two form.

Suppose that this action is invariant in a background with N commuting isometries (abelian duality). For an exhaustive discussion see reference [1]. The Abelian case corresponds to $N = 1$. This isometry could be spacelike or timelike. The set of N isometries forms the isometry group and constitutes a global symmetry of the bosonic string action. The isometry on the direction of X^θ is given by: $X^\theta \rightarrow X^\theta + \omega$. So, we can adapt coordinates $X^I = (X^\theta, X^i)$ with $i = 1, \dots, D - 1$, such that the isometry acts by the transformation of X^θ , and the background fields (G, B, Φ) are independent of X^θ . The duality algorithm implies (in the

conformal gauge) that:

$$\hat{S}_\theta = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \left[(G_{IJ} + B_{IJ}) \partial X^I \bar{\partial} X^J + (J_k - \partial\lambda) \bar{A} + (\bar{J}_k - \bar{\partial}\lambda) A + k^2 A \bar{A} \right]. \quad (2)$$

Making the substitution of this action onto the path integral and integrating over λ , we recover the original action. Applying the last step of the algorithm we get the dual theory. Performing an integration with respect to X, A and \bar{A} we find the dual action \tilde{S} , which has an identical form than \hat{S} . This and the condition that Z must be the same determines that the dual background fields $(\tilde{G}, \tilde{B}, \tilde{\Phi})$ will be related with the original ones by the *Buscher transformations*:

$$\begin{aligned} \tilde{G}_{\theta\theta} &= \frac{1}{G_{\theta\theta}}, & \tilde{G}_{\theta i} &= \frac{B_{\theta i}}{G_{\theta\theta}}, \\ \tilde{G}_{ij} &= G_{ij} - \frac{G_{\theta i} G_{\theta j} - B_{\theta i} B_{\theta j}}{G_{\theta\theta}}, & \tilde{\Phi} &= \Phi - \frac{1}{2} \ln G_{\theta\theta}, \\ \tilde{B}_{ij} &= B_{ij} + \frac{G_{\theta i} B_{\theta j} - B_{\theta i} G_{\theta j}}{G_{\theta\theta}}, & \tilde{B}_{\theta i} &= \frac{G_{\theta i}}{G_{\theta\theta}}, \end{aligned} \quad (3)$$

where the expression of the dual dilaton is a consequence of the conformal invariance preservation. These equations of transformation could be easily generalized to the case of N commuting isometries. That case is studied in [1]. The general procedure for the dualization of any field theory was proposed by Rocek and Verlinde [9]. The extension to the non abelian case was proposed by de la Ossa and Quevedo in [10] and it was lately studied by many authors. T duality was originally discovered in toroidal compactifications finding that interchanges winding states by momentum states (Kaluza-Klein) states in the compactified theory. Buscher duality is a generalization of T-duality; it is contained in eqs. (3) that X^θ is a compact spacelike coordinate, that means, when X^θ direction could be identified with S^1 . So T duality is just a particular case of Buscher duality which exists provided that there is a global isometry. When the isometry is timelike and the coordinate is non compact we are in the limit case $R_0 \rightarrow \infty$.

This paper is organized as follows: In subsection 2.1 we establish the equations of motion for the background fields in the low energy effective action. Subsection 2.2 deals with the case in which Buscher duality is along a timelike isometry. In section 3 we extend the symmetries of the Bosonic String. The subsections 3.1 and 3.2 introduce the so called Wick duality and summarize the procedure we follow for the composition of the dualities used in this paper. Finally in section 4 we give our concluding remarks.

2.1 BTZ Black Hole in String Theory

Now we will survey the BTZ black hole solution in $2+1$ GR and its embedding into string theory in three dimen-

sions. In the context of the effective string action in three dimensions, we will obtain Buscher duality along a timelike isometry and we will discuss some consequences of this solution. For simplicity we will restrict ourselves to the non-rotating solution, i.e. $J = 0$. In three dimensions, the effective low energy action for strings is given by:

$$S = \int d^3x \sqrt{-g} e^{-2\Phi} \left(\frac{4}{k} + R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 \right) + \mathcal{O}(\alpha'^2) \quad (4)$$

where the axion $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ is a totally antisymmetric tensor field ($\mu, \nu = 0, 1, 2$) and we used the shorthand $H^2 = H_{\mu\nu\rho}H^{\mu\nu\rho}$. The equations of motion are:

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4}H_{\mu\lambda\sigma}H_\nu^{\lambda\sigma} = 0, \quad (5)$$

$$\nabla^\sigma (e^{-2\Phi} H_{\sigma\mu\nu}) = 0, \quad (6)$$

$$4\nabla^2\Phi - 4(\nabla\Phi)^2 + \frac{4}{k} + R - \frac{1}{12}H^2 = 0. \quad (7)$$

Now we will try to find a particular solution for these field equations. For the metric we choose the well known [24] BTZ black hole

$$ds_{BTZ}^2 = -\left(\frac{r^2}{l^2} - M\right) dt^2 + \left(\frac{r^2}{l^2} - M\right)^{-1} dr^2 + r^2 d\phi^2 \quad (8)$$

where M is identified with the mass parameter, l is a constant with dimensions of length (called the cosmological radius) which is related to the cosmological constant by $\Lambda = -\frac{1}{l^2}$. The BTZ black hole metric is a particular solution of the background field equations Eqs.(5, 6, 7).

In the simplest case the dilaton field can be taken constant, $\Phi = \text{constant}$. Since in three dimensions $H_{\mu\nu\rho}$ is proportional to the volume three form $\varepsilon_{\mu\nu\rho}$, we can see that the solution to the equations of motion is reduced to find a proportionality constant between $H_{\mu\nu\rho}$ and $\varepsilon_{\mu\nu\rho}$. The solution to the background field equations is

$$g_{\mu\nu}^{BTZ}, \quad H_{\mu\nu\rho} = \frac{2}{l} \varepsilon_{\mu\nu\rho} \quad \text{and} \quad \Phi = \text{const.} \quad (9)$$

where $g_{\mu\nu}^{BTZ}$ is defined by $ds_{BTZ}^2 = g_{\mu\nu}^{BTZ} dx^\mu dx^\nu$.

Now let us remind an important result of Horowitz and Welch [25]:

“Every solution to Einstein field equations in three dimensions with a negative cosmological constant, is a solution to low energy string theory with $\Phi = \text{const}$ and $H_{\mu\nu\rho} = \frac{2}{l} \varepsilon_{\mu\nu\rho}$ ”.

2.2 Dual BTZ Metric Along a Timelike Isometry

We have seen that given a solution $(g_{\mu\nu}, B_{\mu\nu}, \Phi)$ of the field equations, which is independent of a coordinate, let's say x^θ , there exists another Buscher dual solution $(\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\Phi})$. We are interested in the case $x^\theta = x^0 = t$ (timelike isometry), so applying the Buscher transformation to the solution given before Eq. (9) we get

$$\tilde{g}_{tt} = \left(\frac{r^2}{l^2} - M\right)^{-1}, \quad \tilde{g}_{t\phi} = \frac{r^2}{l} \left(\frac{r^2}{l^2} - M\right)^{-1},$$

$$\tilde{\Phi} = \text{const.} - \frac{1}{2} \ln \left| \frac{r^2}{l^2} - M \right|, \quad \tilde{B}_{\mu\nu} = 0,$$

$$\tilde{g}_{rr} = \left(\frac{r^2}{l^2} - M\right)^{-1}, \quad \tilde{g}_{\phi\phi} = -\left(\frac{r^2}{l^2} - M\right)^{-1} Mr^2. \quad (10)$$

The dual fields $\tilde{H}_{\mu\nu\rho}$ are obviously zero too. It is a direct procedure to check that the triplet $(\tilde{g}, \tilde{H}, \tilde{\Phi})$ satisfies the background field equations.

3 Extended Symmetries of the Bosonic String in 3 Dimensions: BTZ solution

For T-duality, a string does not distinguish the geometry of the background in which it is moving; nevertheless, in the frame of GR those geometries could be totally different. In the following sections we will go beyond the previous results by introducing a new “duality” in string theory (at low energy effective action in 2+1 dimensions) called *Wick Duality*.

Wick Duality was proposed and used in GR in Ref. [23]. This duality consists in an identification between a ‘big-bang’ cosmological solution and an Euclidean BTZ metric (with a negative cosmological constant). This kind of duality is relevant in GR in 2+1 dimensions because here we have a topological theory and so it is independent of the metric (see Ref. [22, 23]). Thus Wick duality provides us two different (dual) descriptions of the same theory. Both descriptions must have, at least, different signature.

Consider for example the Euclidean BTZ solution with mass $M \geq 0$ and vanishing angular momentum

$$ds_{Euc}^2 = \frac{(r^2 - Ml^2)}{l^2} d\tau^2 + \frac{l^2}{r^2 - Ml^2} dr^2 + r^2 d\phi^2. \quad (11)$$

It is an easy matter to see that this metric is a solution of the Einstein equations with non-zero cosmological constant, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. The scalar curvature is given by $R = -6/l^2$ and therefore $-\frac{1}{l^2} = \Lambda < 0$. Now we define *Wick duality* and apply it to the above BTZ solution. The Wick duality is defined as:

$$\text{BTZ} \xrightarrow{\mathcal{W}} \widehat{\text{BTZ}} \quad (12)$$

where \mathcal{W} represents the Wick duality transformation which is given by the following rotations:

$$\tau \rightarrow i\tau, \quad l \rightarrow il. \quad (13)$$

Thus the BTZ Euclidean metric transforms to:

$$\widehat{ds}_{Euc}^2 = -\frac{l^2}{\mathcal{T}^2 + Ml^2} d\mathcal{T}^2 + \frac{(\mathcal{T}^2 + Ml^2)}{l^2} d\mathcal{R}^2 + \mathcal{T}^2 d\phi^2 \quad (14)$$

where we have renamed coordinates $r \rightarrow \mathcal{T}$ and $\tau \rightarrow \mathcal{R}$. The last equation can be interpreted as the metric describing a “cosmological Big-Bang”. The Ricci scalar for this space-time is given by $R = 6/l^2$ which corresponds to $\Lambda > 0$.

3.1 Wick Duality as a Symmetry of the Bosonic String Action

It needs to be remarked that Wick duality doesn't represent (*a priori*) a symmetry of string theory, and consequently we should not expect that Wick-dual solutions satisfy the background field equations for the bosonic string.

On the other hand, the sets of solutions like $(\widetilde{g}, \widetilde{B}, \widetilde{\Phi})$, obtained via Buscher duality Eqn.(3), are constructed *ad hoc* to satisfy the background field equations.

Here we propose a possible extension of the solutions of the background fields for the bosonic string.

Let's start by the Euclidean action for the bosonic string:

$$S_E = \int d^3x \sqrt{g} e^{-2\Phi} \left(-\frac{4}{k} - R - 4(\nabla\Phi)^2 + \frac{1}{12} H^2 \right), \quad (15)$$

where $\mu, \nu = 1, 2, 3$. This action is the result of a Wick rotation on time $t \rightarrow -it$ in the usual Lorentzian action Eq.(4). In this case, the Euclideanized action S_E is given by $S_E = -iS_L$ where S_L is the regular Lorentzian action. The equations of motion for the Euclideanized action are the same as those given in Eqs.(5, 6, 7). A more detailed discussion can be found in Ref. [17].

Now we are going to prove that the multiplet obtained through Wick Duality $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ —in which \widehat{g} will correspond to a metric for a three dimensional spacetime with positive cosmological constant— is a solution of the background field equations in the Lorentz regime.

Consider again the BTZ metric given in eq.(8). When Wick duality is applied to this solution (and after renaming coordinates $r \rightarrow \mathcal{T}$ and $\tau \rightarrow \mathcal{R}$), we get the dual metric

$$\widehat{ds}^2 = -\left(M + \frac{\mathcal{T}^2}{l^2}\right)^{-1} d\mathcal{T}^2 - \left(M + \frac{\mathcal{T}^2}{l^2}\right) d\mathcal{R}^2 + \mathcal{T}^2 d\phi^2 \quad (16)$$

for which the scalar curvature is $\widehat{R} = 6/l^2$, so we have a space with a positive cosmological constant. Thus, we have

obtained a new solution of Einstein field equations through a transformation which is not a string symmetry. The question that naturally arises at this point is: Does the corresponding triplet $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ satisfy the background field equations for the bosonic string? And, if that is the case: What happens if we consider a new T-duality transformation for $(\widehat{g}, \widehat{B}, \widehat{\Phi})$? The rest of the (Wick dualized) gravitational multiplet is given by:

$$\widehat{H}_{\mu\nu\rho} = \frac{2}{l} \widehat{\varepsilon}_{\mu\nu\rho} \quad \text{and} \quad \widehat{\Phi} = \text{const.} \quad (17)$$

It is a straightforward calculation to verify that Eqs.(16, 17) satisfy the Eqs.(5, 6, 7) and consequently the multiplet $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ is a solution of the background fields equations for the bosonic string.

The corresponding ‘duality’ for gravity in (2+1) dimensions (i.e for the metric \widehat{g}) is explored in detail in Ref. [23]. In string theory, however, both solutions (g, B, Φ) and $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ represent dual solutions of the same background field equations.

3.2 Properties of the Wick's Duality for Bosonic String

In the preceding sections we have seen some particular examples of the solutions of the string background and the Buscher duality is acting on those solutions. In addition we have seen that Wick duality provides a relation between different solutions of Einstein field equations.

Let us summarize the procedure we will follow for the study of the properties of Wick duality. We will see that the following diagram:

$$\begin{array}{ccc} \text{String}_{\text{BTZ}} & \xrightarrow{\mathcal{W}} & \text{String}_{\widehat{\text{BTZ}}} \\ \downarrow \mathcal{B} & & \downarrow \mathcal{B} \\ \text{String}_{\widetilde{\text{BTZ}}} & \xrightarrow{\mathcal{W}} & \text{String}_{\widetilde{\widehat{\text{BTZ}}}} \end{array} \quad (18)$$

is, in general a commutative diagram, and we will describe the consequences of this fact. Vertical arrows stand for the action of Buscher duality \mathcal{B} and horizontal arrows stand for the operation of Wick Duality \mathcal{W} in strings (it includes the fields (g, B, Φ)). Here, the hat symbol $\widehat{}$ denotes the Wick duality mapping, i.e. $\mathcal{W}(\text{String}_{\text{BTZ}}) \equiv \text{String}_{\widehat{\text{BTZ}}}$. On the other hand the symbol $\widetilde{}$ is used to denote the Buscher duality mapping i.e. $\mathcal{B}(\text{String}_{\text{BTZ}}) \equiv \text{String}_{\widetilde{\text{BTZ}}}$.

We have seen how $\text{String}_{\text{BTZ}}$ transforms when we perform a timelike Buscher duality (see for instance Eq.(10)). The result may be summarized as

$$\begin{aligned} \widetilde{ds}^2 &= N(r)^{-2} \left(-d\tau^2 + dr^2 + r^2 N(r)^2 d\phi^2 \right), \\ \widetilde{H}_{\mu\nu\rho} &= 0 \quad \text{and} \quad \widetilde{\Phi} = \text{const.} - \frac{1}{2} \ln \left| \frac{r^2}{l^2} - M \right|, \end{aligned} \quad (19)$$

where we used the notation $d\tau \equiv dt - (r^2/l)d\phi$ and $N^2(r) \equiv (r^2/l^2 - M)$. This triplet $(\tilde{g}, \tilde{B}, \tilde{\Phi})$ is what we denote by $\text{String}_{\widetilde{\text{BTZ}}}$.

Let us go back to the diagram (18). We can see that it is necessary to calculate the operation: $\text{String}_{\widetilde{\text{BTZ}}} \xrightarrow{\mathcal{W}} \text{String}_{\widetilde{\text{BTZ}}}$. It is convenient to remember that the dimension of the background space is (2+1) and that the duality operations suggest that it is possible to extend the string symmetries, obtaining in this way new solutions to the background field equations. However, we cannot expect that this be the case for gravity in (3+1) dimensions.

For the metric of the triplet $\mathcal{W}(\text{String}_{\widetilde{\text{BTZ}}})$ we obtain (whit the new choice of coordinates $r \rightarrow \mathcal{T}$ and $\tau \rightarrow \mathcal{R}$)

$$\widehat{ds}^2 = \left(\frac{\mathcal{T}^2}{l^2} + M \right)^{-1} \left(\begin{array}{l} - d\mathcal{T}^2 - d\mathcal{R}^2 \\ + \mathcal{T}^2 \left(\frac{\mathcal{T}^2}{l^2} + M \right) d\phi^2 \end{array} \right) \quad (20)$$

where we defined $d\mathcal{R}' \equiv (d\mathcal{R} + \frac{\mathcal{T}^2}{l} d\phi)$. Let us see how the other fields transform. For the field $\tilde{B}_{\mu\nu}$, Wick duality does not introduce new components, then $\tilde{B}_{\phi t} = 0 = \widehat{B}_{\phi\mathcal{R}}$. The rest of the dual fields are

$$\widehat{H}_{\mu\nu\rho} = 0 \quad \text{and} \quad \widehat{\Phi} = \text{const.} - \frac{1}{2} \ln \left| \frac{\mathcal{T}^2}{l^2} + M \right|. \quad (21)$$

Once again, we can verify that the new triplet $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ is a solution of the background field equations. The new Kalb-Ramond field is zero (and consequently the axion field as well) and then it is not relevant if the background field equations are referred to a Lorentzian or Euclidean regime.

In reference to the commutative diagram (18) we conclude that the composition of Buscher and Wick dualities give us the same dual solutions (20, 21) of the background field equations. It seems that the ‘‘price’’ we have to pay for using Buscher duality along a timelike coordinate, is the appearance of the signature $(-, -, +)$ for the (Wick)dual metrics. However, from the point of view of string theory the procedure is completely well defined.

4 Conclusions

For the case of the effective action of Strings in the Lorentz frame we have studied some properties of Wick duality \mathcal{W} , in particular we have shown that Wick duality and Buscher duality \mathcal{B} —along a time-like isometry— commute. This means that String Wick duality is consistent with Buscher duality and therefore with the theory of Strings in three dimensions. Then we have extended the symmetry

provided by Buscher duality connecting a wider set of solutions of the effective action of String Theory.

During the process we have used the well known BTZ solution of 2+1 dimensional General Relativity with cosmological constant. There could be causality problems in the BTZ solution for small radius R_0 . This appears because we used a T-duality transformation along a time-like isometry. However it is not known if String Theory has an analog of the chronological censorship conjecture at Plank’s scale.

However we can still use T-duality in the case of topological field theories. [17, 18]. In this sense Chern-Simons theory of General Relativity in 2+1 dimensions has a remarkable importance [14]. Then time-like T-duality is well defined and represents a genuine symmetry for this theory and its topological extensions —as we have seen in last section—. This can be seen as another use of time-like T-duality, besides the well known applications obtaining *de Sitter* spaces from String Theory [19]. Nevertheless as we just mentioned T-duality affects the causal structure of BTZ space-time in General Relativity. We have conjectured also that due to the topological structure of three dimensional String Theory, we could extend String Wick duality to all orders in α' in the perturbative theory. In this paper we have considered solutions to the metric with $J = 0$. If $J \neq 0$ the results we have calculated do not change. There is no agreement [15] about the convenience of considering a charged BTZ BH.

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References

- [1] A. Giveon and D. Kutasov, *Rev. Mod. Phys.* **71** (1999) 983, hep-th/9802067; A. Sen, ‘‘An Introduction to Non-perturbative String Theory’’, hep-th/9802051.
- [2] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, ‘‘An Introduction to T-duality in String Theory’’, *Nucl. Phys. Proc. Suppl.* **41** (1995) 1; F. Quevedo, ‘‘Duality and Global Symmetries’’, *Nucl. Phys. Proc. Suppl.* **61 A** (1998) 23.
- [3] A. Achúcarro and P.K. Townsend, *Phys. Lett. B* **180** (1986) 89; E. Witten, *Nucl. Phys. B* **311** (1988) 46; **323** (1989) 113.
- [4] E. Witten, *Commun. Math. Phys.* **121** (1989) 351.
- [5] A.P. Balachandran, L. Chandar and B. Sathiapalan, *Int. J. Mod. Phys. A* **11** (1996) 3587.

- [6] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, “Remarks on $2 + 1$ Self-dual Chern-Simons Gravity”, *Phys. Rev. D*, **61** (2000) 0850022-1.
- [7] E. Witten, “Chern-Simons Gauge Theory as a String Theory”, hep-th/9207094.
- [8] T. Buscher, *Phys. Lett. B* **191** (1987) 59; *B* **201** (1988) 166.
- [9] M. Roček and E. Verlinde, *Nucl. Phys. B* **373** (1992) 630.
- [10] X.C. de la Ossa and F. Quevedo. “Duality Symmetries from Non-abelian Isometries in String Theory”, *Nucl. Phys. B* **403** (1993) 377.
- [11] E. Alvarez, L. Alvarez-Gaumé, J.L.F. Barbón and Y. Lozano, “Some Global Aspects of Duality in String Theory”, *Nucl. Phys. B* **415** (1994) 71, hep-th/9309039.
- [12] A. Giveon and M. Roček, “On Non-abelian Duality”, *Nucl.Phys. B* **421** (1994) 173, hep-th/9308154.
- [13] E. Alvarez, L. Alvarez-Gaumé, and Y. Lozano, “On Non-abelian Duality”, *Nucl. Phys. B* **424** (1994) 155, hep-th/9403155.
- [14] S. Carlip, *Quantum Gravity in 2+1 Dimensions*, Cambridge University Press, (1998); M. Bañados, “Notes on Black Holes and Three Dimensional Gravity”, hep-th/9903244.
- [15] A. García, “On Rotating Charged BTZ Metric”, hep-th/9909111.
- [16] C.Martínez, C. Teitelboim and J. Zanelli, “Charged Rotating Black Hole in Three Spacetime Dimensions”, *Phys.Rev. D* **61** (2000) 104013, hep-th/9912259.
- [17] C.M. Hull, “Duality ans Strings, Space and Time”, hep-th/9911080.
- [18] C.M. Hull, “De Sitter Space in Supergravity and M Theory”, hep-th/0109213.
- [19] C.M. Hull, “Timelike T-duality, de Sitter space, large N gauge theories and topological field theory”, *JHEP07(1998)021* .
- [20] H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido, “On S -duality in $(2+1)$ -Chern-Simons Supergravity”, *Phys. Rev. D* **64** (2001) 024002-1.
- [21] E. Witten, “The Central Charge in Three Dimensions”, *Physics and Mathematics of Strings* eds. L. Brink, D. Friedan and A.M. Polyakov, World Scientific (1990).
- [22] E. Witten, “Quantization of Chern-Simons Gauge Theory with Complex Gauge Group”, *Commun. Math. Phys.* **137** (1991) 29.
- [23] A. Corichi and A. Gomberoff, “On a Spacetime Duality in $(2 + 1)$ -dimensions”, *Class. Quant. Grav.* **16** (1999) 3579, gr-qc/9906078.
- [24] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* **69** (1992) 1849.
- [25] G.T. Horowitz and D.L. Welch, “Exact Three Dimensional Black Holes in String Theory”, *Phys. Rev. Lett.* **71** (1993) 328, hep-th/9302126.