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Spontaneous Emission: Validity Of The Secular Approximation For The Rabi Model With 1 And 2 Atoms

Emisión Espontánea: Validez De La Aproximación Secular Para El Modelo De Rabi Con 1 y 2 Átomos

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Resumen

En este artículo estudiamos la validez de la aproximación secular para el estudio de $N = 1, 2$ átomos que interactúan resonantemente con un campo de radiación clásico y con el vacío electromagnético. En particular, comparamos los resultados obtenidos de la aproximación secular con la solución exacta del problema para 1 y 2 átomos. Encontramos que la aproximación es válida para describir la posición y los anchos de los picos del espectro. Sin embargo, también mostramos que la forma del operador densidad estacionario del sistema no es adecuadamente descrita en dicha aproximación.

Palabras Clave: Intensidades y formas de las líneas espectrales atómicas; Espectros atómicos.

Abstract

In this article we study the validity of the secular approximation for the study of $N = 1, 2$ atoms interacting resonantly with a classical radiation field and the electromagnetic vacuum. In particular we compare the results obtained using the secular approximation with the exact solution for 1 and 2 atoms. We find that the approximation is valid for the description of the positions and widths of the peaks of the spectrum. Nevertheless, we also show that the stationary density matrix of the system is not well described in the secular approximation.

Keywords: Intensities and shapes of atomic spectral lines; Atomic spectra.
PACS: 32.70.-n; 32.30.-r.

1 Introduction

Integrable models are a valuable tool for the study of physical systems. Models that are analytically solvable are valuable since the interaction of the parameters involved in the model is seen very neatly once a closed form expression is at hand. In this paper we study the validity of the approximations used to obtain one of those analytically solvable models, a system of N atoms interacting with a classical radiation field and the electromagnetic vacuum. In particular, we are interested in study the validity of the secular approximation for the system described above.

In section 2 the secular approximation is introduced and the secular master equation is derived. In section 3 we compare the solution obtained in the secular approximation with the exact solution of the model. In section 4 the same analysis is done but for the case of two atoms. Finally, the conclusions are presented in section 5

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2 The Secular Approximation for N atoms

In order to introduce the secular approximation let us study a system of N two level atoms interacting resonantly with the classical radiation field and the electromagnetic vacuum. The Master equation governing the time evolution of the density operator of the system is [1, 2]:

$$\begin{aligned}\frac{d\rho}{dt} &= i[H,\rho] + \Gamma\mathcal{L}\{\rho\} \\ &= i\omega[S^+ + S^-, \rho] \\ &\quad -\Gamma(S^+S^-\rho - 2S^-\rho S^+ + \rho S^+S^-),\end{aligned}$$

where $S^{\pm}$ are the angular momentum ladder operators [3] ($S = \frac{N}{2}$), ω is the Rabi frequency and $\Gamma/2$ is the Einstein A coefficient [4]. In the last equation both the rotating wave approximation (RWA) and the Born-Markov approximation have been made [2, 4].

We are interested in writing the liouvillian superoperator $\mathcal{L}\{\rho\}$ as a sum of terms involving fast and slow operators, that is operators that in the Heisenberg picture and without dissipation evolve with 0 frequency (slow operators) and with frequencies 2ω and 4ω (fast operators). To see this more neatly the angular momentum operators can re-labeled as follows:

$$S_y \to R_x, \quad S_z \to R_y, \quad S_x \to R_z. \tag{1}$$

In terms of the R operators the master equation is:

$$\frac{d\rho}{dt} = i2\omega[R_z,\rho] - \Gamma\left(\mathcal{L}_0\{\rho\} + \mathcal{L}_1\{\rho\} + \mathcal{L}_2\{\rho\}\right), \tag{2}$$

$$\begin{aligned}\mathcal{L}_0\{\rho\} &= [R_z,[R_z,\rho]] + \\ &\quad \frac{1}{4}\left([R^+,[R^-,\rho]] + [R^-,[R^+,\rho]]\right), \\ \mathcal{L}_1\{\rho\} &= -\frac{i}{2}\left\{[R_z, R^+ + R^-],\rho\right\} \\ &\quad -\frac{i}{2}\left((R^+ + R^-)\rho R_z - R_z\rho(R^+ + R^-)\right), \\ \mathcal{L}_2\{\rho\} &= \frac{1}{4}\left\{R^+R^+ + R^-R^-,\rho\right\},\end{aligned}$$

where $\{A,B\} \equiv AB + BA$ and $[A,B] \equiv AB - BA$.

For strong fields ($\omega \gg \Gamma$) it is reasonable to drop the terms $\mathcal{L}_1\{\rho\}$ and $\mathcal{L}_2\{\rho\}$ and to retain only the slowly varying part $\mathcal{L}_0\{\rho\}$. Notice that in the Heisenberg picture with $\Gamma = 0$ the operators evolve according to $R^{z(0)}(t) = R^{z(0)}(0)$ and $R^{\pm(0)}(t) = R^{\pm(0)}(0)e^{\pm 2i\omega t}$. The terms involved in $\mathcal{L}_0\{\rho\}$ evolve with 0 frequency, whereas the terms in $\mathcal{L}_1\{\rho\}$ and $\mathcal{L}_2\{\rho\}$ evolve with frequencies $\pm 2\omega$ and $\pm 4\omega$ respectively.

In reference [1] it is explained that corrections to the secular approximation will be of the order $\left(\frac{\Gamma N}{\omega}\right)^2$. Under the secular approximation it is possible to obtain a closed set of differential equations (and an exact solution) for the expectation values of the operators S_x, S_y and S_z for any number of atoms. One can also show that the asymptotic state of the system is given by the density operator: $\rho_{secular}(\infty) = \frac{1}{N+1}\hat{1}$ (an infinite temperature state). [1].

3 The Secular Approximation for one Atom

For one atom all the information contained in the density matrix is equivalent to the information contained in the *spherical* components of the Bloch vector $\vec{v} = (\langle S_z\rangle, \langle S^+\rangle, \langle S^-\rangle)$. The Bloch vector satisfies a differential equation in both the secular approximation and in the exact master equation of the form $\frac{d}{dt}\vec{v} = A\vec{v} + \vec{f}$. In the secular approximation the eigenvalues of the matrix A (which are related to the widths and positions of the peaks of the spectrum) are given by:

$$Spec(A_{secular}) = \left\{-\Gamma, -\frac{3}{2}\Gamma \pm 2i\omega\right\}, \tag{3}$$

whereas in the exact master equation are:

$$Spec(A_{exact}) = \left\{-\Gamma, -\frac{3}{2}\Gamma \pm 2i\omega\sqrt{1 - \frac{\Gamma^2}{16\omega^2}}\right\}. \tag{4}$$

Notice that at $\Gamma = 4\omega$ the system becomes overdamped, that is, all the eigenvalues become purely real.

For the exact master equation one can also obtain the asymptotic state of the system, in this case given by:

$$\rho_{exact}(\infty) = \begin{pmatrix} \frac{1}{2+\left(\frac{\Gamma}{2\omega}\right)^2} & \frac{i\frac{\Gamma}{2\omega}}{2+\left(\frac{\Gamma}{2\omega}\right)^2} \\ \frac{-i\frac{\Gamma}{2\omega}}{2+\left(\frac{\Gamma}{2\omega}\right)^2} & 1 - \frac{1}{2+\left(\frac{\Gamma}{2\omega}\right)^2} \end{pmatrix}. \tag{5}$$

The last matrix is written in the basis of eigenstates of S_z and S^2. From equation (5) it is seen that the corrections to the coherences of the density matrix are *not* of second order in $\frac{\Gamma}{\omega}$.

Finally, one can compare the first correlation function in the secular approximation with the exact solution. By comparing the exact solution [5] with the approximate solution [1] it is seen that the first order effects that we encountered in the stationary density operator are also present in the first order correlation function.

4 The Secular Approximation for two Atoms

For two atoms the information contained in the density matrix is equivalent to the information contained in the expectation values of the operators:

$$\vec{q} = \langle\{S_x, S_y, S_z, S_x^2, S_y^2, S_z^2, S_{yz}, S_{xy}, S_{zx}\}\rangle,$$

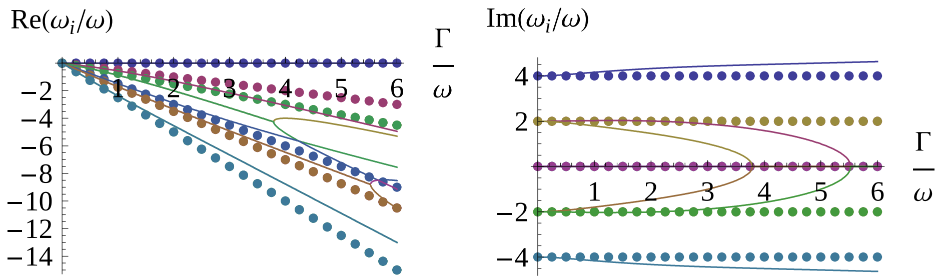

Fig. 1: Comparison between the eigenvalues of the matrix A for the exact (continuous line) and secular (points) master equation. It is seen that when $\Gamma/\omega \ll 1$ both descriptions are in excellent agreement.

where we have defined:

$$S_{ij} \equiv \frac{S_i S_j + S_j S_i}{2}.$$

As in the case of one atom the vector $\vec{q}$ satisfies a differential equation of the form $\frac{d}{dt}\vec{q} = A\vec{q}$. When the secular approximation is used the real matrix A takes a block diagonal form of blocks 1×1 , 2×2 , 2×2 and 4×4 (which allows to solve the problem in closed form). The comparison between the eigenvalues of the matrix A obtained from the master equation and the ones obtained with the secular approximation is shown in Figure 1. It is seen that the imaginary parts of the eigenvalues evolve approximately as 0, $\pm 2\omega$ and $\pm 4\omega$ which are precisely the frequencies of the slow and fast operators in the Heisenberg picture and that in the secular approximation they do not depend on Γ. Finally, notice that at $\Gamma = 3.7910\omega$ and $\Gamma = 5.5269\omega$ the real parts of a pair of complex conjugate eigenvalues in the exact master equation become zero and correspondingly their real parts bifurcate.

To show that the corrections to the eigenvalues obtained in the secular approximation are actually of second order in Γ/ω one can calculate the following limit

$$\delta = \lim_{\frac{\Gamma}{\omega}\to 0} \frac{|Spec(A_{exact}) - Spec(A_{secular})|}{\left(\frac{\Gamma}{\omega}\right)^2} = 0.6147\omega, \tag{6}$$

which shows that the differences between the eigenvalues of the matrix A in the secular and exact master equation are indeed of second order in Γ/ω (The limit δ defined in the last equation can be easily obtained for the case of one atom using equations (3), (4): $\delta = \sqrt{2}\omega/16$).

One can also obtain the stationary density matrix in the exact master equation:

$$\rho_{exact}(\infty) = \frac{1}{\Gamma^4 + 4\omega^2\Gamma^2 + 12\omega^4} \times \begin{pmatrix} 4\omega^4 & 2i\sqrt{2}\Gamma\omega^3 & -2\Gamma^2\omega^2 \\ -2i\sqrt{2}\Gamma\omega^3 & 2\omega^2\left(\Gamma^2 + 2\omega^2\right) & i\sqrt{2}\Gamma\omega\left(\Gamma^2 + 2\omega^2\right) \\ -2\Gamma^2\omega^2 & -i\sqrt{2}\Gamma\omega\left(\Gamma^2 + 2\omega^2\right) & \Gamma^4 + 2\omega^2\Gamma^2 + 4\omega^4 \end{pmatrix}.$$

Again it is seen that in the coherences there are contributions of the order Γ/ω.

5 Conclusions

We have studied the validity of the secular approximation for one and two 2-level atoms. We have shown that although the secular approximation is valid, i.e., their corrections are of order $(\Gamma/\omega)^2$, for the description of the values of the observables of one and two atoms the corrections to the approximation for the equilibrium density matrix are of order Γ/ω. The last observation is important because for the calculation of the spectrum of the atomic system not only the dynamical equations are involved but also the asymptotic values of the observables which in turn depend on the form of the stationary density matrix.

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