



# Two-Electron Energy Spectrum in Concentric Quantum Ribbons

## Espectro de energía de dos electrones en cintas cuánticas concéntricas

M. R. Fulla<sup>1\*</sup>, J. H. Marín C<sup>2</sup>.

<sup>1</sup>Institución Universitaria Pascual Bravo A. A. 6564, Medellín.

<sup>2</sup>Escuela de Física, Universidad Nacional de Colombia -Sede Medellín. A. A. 3840

Recibido abril 25 de 2012; aceptado junio 20 de 2011

---

### Resumen

Se estudia el espectro energético de dos electrones obligados a moverse en dos cintas cilíndricas coaxiales. El sistema bi-particular se encuentra inmerso en un campo magnético homogéneo orientado a lo largo del eje de simetría común. Mediante el uso de coordenadas relativas y de centro de masa, se obtiene una ecuación diferencial unidimensional de segundo orden, la cual se resuelve numéricamente usando la técnica de barrido trigonométrico. Los resultados del cálculo del espectro energético bi-particular, se presentan como función del campo magnético y de la razón entre los radios de las cintas para valores diferentes de la altura. En los casos límites en que las cintas se transforman en anillos unidimensionales, nuestros resultados coinciden con los reportados previamente para partículas en espacios unidimensionales.

**Palabras claves:** cintas cuánticas, efecto Aharonov-Bohm, aproximación adiabática, sistema bi-electrónico

### Abstract

We study the energy spectrum of two-electron constrained to move in two concentric quantum ribbons under a homogeneous magnetic field oriented along the common symmetry axis. By using the center of mass and relative coordinates, we obtain a second order one-dimensional equation which is numerically solved by means of the trigonometric sweep method. The results of calculations for the two-electron energy spectrum are presented as a function of the ratio between the quantum ribbon radii, at different heights, and the magnetic field. In the limit case when both radii become equal, and the height of the quantum ribbons tends to zero, our results coincides with those previously reported.

**Keywords:** quantum ribbons, Aharonov-Bohm effect, adiabatic approximation, two-electron system

© 2009 Revista Colombiana de Física. Todos los derechos reservados.

---

## 1. Introduction

Study of energy spectrum of particles confined in quantum systems with cylindrical symmetry has generated a great deal of attention since the first reports of carbon nanotubes [1]. Recently, V. Prinz [2,3] has experimentally established the possibility of fabricating semiconductor nanotubes, which can be obtained from two different semiconductors with a lattice parameter mismatch up to 7%. This discontinuity inextricably yields strain effects during the epitaxial growth, in such way that an overgrowth can lead to a stress relaxation due to a reduction of the elastic energy. In consequence, the structure spontaneously is rolled-up over itself

adopting a tube shape. This semiconductor structure in a tube shape is characterized by its thickness; height and diameter are strongly dependent on the initial semiconductor bilayer width as well as the stress mechanism generated as a consequence of the mismatch parameter. The crucial point of this report is related to the possibility to fabricate quasi-bidimensional cylindrical ribbons by tailoring their geometrical parameters by means of a controlled growth process. These semiconductor nanotubes display remarkable features in comparison with those made from carbon. For instance, the semiconductor nanotubes have a diameter much greater than the carbon nanotubes and its curvature is

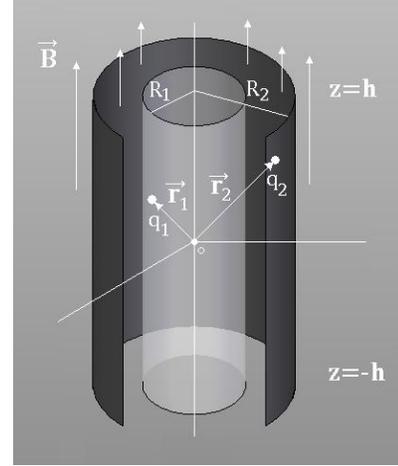
more regular. In this sense, the cylindrical semiconductor nanostructures become more suitable for potential applications due to the strong relation between nanotube diameter and “magnetic field length” [4-5] which allows us to achieve a great control on the carrier quantum states. This fact has been demonstrated both theoretically and experimentally [4] by considering only one electron into the single semiconductor nanotube [5]. The results show fundamental differences related to electrical current distributions generated in cylindrical and planar structures. Despite the importance of this study [5], we also consider interesting to analyze the case of two particles constrained to move into two dimensional cylindrical ribbons, since this allows us to study the electron-electron correlation, and the influence on energy spectrum in two dimensional spaces. We are aware that energy structure of a two-electron system is easier to analyze in strictly one dimensional quantum rings [6-7], since electron-electron energy spectrum can be obtained exactly. Nevertheless, it is necessary to emphasize that one-dimensional systems are restrictive because the electrons can never have the same angular position [6], since they are constrained to rotate around the perpendicular symmetry axis. Additionally, in these one dimensional rings [6] is not possible to study attractive interactions between particles with different charge such as an electron-hole pair due to the Coulomb interaction in one dimensional space is unstable. This difficulty is overcome if the particles are confined in a two dimensional space, similar to a ribbon with cylindrical symmetry. For this reason, the cylindrical ribbons can become more appropriated structures in the purpose of establishing the electronic properties of two particle systems in a two-dimensional space. At the same time these cylindrical ribbons allows us to explore the influence of the geometrical parameters on the energy structure of a two-particle system. Finally, the knowledge of the electronic properties, and their control are crucial subjects in the purpose of realizing both technological applications, and studies of mesoscopic physics [8], such as the Aharonov-Bohm effect and persistent currents. Motivated by finding new physical effects in two dimensional ribbons with cylindrical symmetry, in this work, we propose to analyze the energy spectrum of two particles, spatially separated in two cylindrical quantum ribbons. Nanostructures concentrically coupled, have been previously studied [9] in ring-shaped quantum dots in which their thickness, and height are very small in comparison with their average radii.

## 2. Theoretical Model

The model considered consists, essentially, of two electrons which are constrained to move into two coaxial cylindrical ribbons, where the common axis is in  $z$ -direction. Both cylindrical shells have the same height, which is equal to  $h$  and inner and outer radius are  $R_{in} = R$ , and  $R_{out} = \alpha R$ , respectively, being  $\alpha$  a dimensionless parameter which

allows us to consider the ratio between the radii of the ribbons on the energy spectrum. When the ratio of outer to inner radii tends to one ( $\alpha \rightarrow 1$ ), and the ribbon's height tends to zero ( $h \rightarrow 0$ ), our results should coincide with those previously reported in the Ref. [6] and Ref. [10] for two particles in a single one dimensional ring, and two concentric rings, respectively.

A schematic picture of two particles in a two concentric ribbons, under the homogeneous magnetic field oriented along the symmetric axis  $z$ , is shown in the figure 1.



**Fig.1.** Schematic picture of two-particle in two coaxial nano-ribbons under the homogeneous magnetic field.

By using the effective-mass approximation, and the same notation used in Ref. [6], the dimensionless renormalized Hamiltonian for the system under analysis can be written in cylindrical coordinates  $\{\vec{r}_i(\rho_i, \theta_i, z_i), i = 1, 2\}$  as:

$$\tilde{H} = \hat{H}R^2 = \sum_{i=1}^2 \left[ -\Delta_i - iy \frac{\partial}{\partial \varphi_i} + \frac{\gamma^2 R_i^2}{4} \right] R^2 + V(|\vec{r}_{12}|)R^2. \quad (1)$$

Here  $\vec{r}_{12}$  is the relative carrier position vector, and  $V(|\vec{r}_{12}|)R^2 = \tilde{U}(|\vec{r}_{12}|)$  is the renormalized Coulomb potential interaction which is given by

$$\tilde{U}(|\vec{r}_{12}|) = \frac{2R}{\sqrt{1 + \alpha^2 - 2\alpha \cos(\varphi_1 - \varphi_2) + ((z_2 - z_1)/R)^2}}, \quad (2)$$

the effective Rydberg  $R_y^* = m_e^* e^4 / 2\hbar^2 (4\pi\epsilon_0\epsilon_r)^2$ , the effective Bohr radius  $a_0^* = \frac{4\pi\epsilon_0\epsilon_r\hbar^2}{m_e^* e^2}$ , and the first Landau level, expressed in  $R_y^*$ ,  $\gamma = \frac{eB\hbar}{2m_e^* c R_y^*}$ , have been taken as units of length, energy, and the dimensionless magnetic field, respectively. In our calculations, we assume nanoribbons made of InGaAs.

In general case, the Hamiltonian (1) is not separable owing to the existence of the Coulomb interaction term in which all coordinates are mixed; therefore, is not possible to obtain exact solutions. For this reason is absolutely necessary to carry out a numerical approach in order to calculate the two-particle energy spectrum. With the aim of solving the eigenvalues problem with Hamiltonian (1), we take into account two different situations. On the one hand, we can assume the ribbons which are characterized by a small radius-to-height aspect ratio, and in the other hand we can assume nanoribbons with small height-to-base aspect ratio, and very narrow width dots similar to those reported in the Ref. [9]. In spite of both cases could be analyzed by using the well-known adiabatic approximation (AA) [11], we prefer to study the second situation due to the possibility to compare the results obtained by us in this research with similar one previously reported for limit cases [6,10]. Under the typical condition,  $h/R \ll 1$ , the AA allows us to decouple fast electron motion in z-direction from the slow electron angular rotation motion around the z-axis. By means of consistent application of the AA, the two-particle Hamiltonian (1) can be written as:

$$\tilde{H} = -\frac{\partial^2}{\partial \varphi_1^2} - \frac{\partial^2}{\alpha^2 \partial \varphi_2^2} - i\gamma \left( \frac{\partial}{\partial \varphi_1} + \tau \frac{\partial}{\partial \varphi_2} \right) + \frac{\gamma^2 R^2 (1 + \alpha^2)}{4} + \bar{U}_{eff}(\varphi_1 - \varphi_2) \quad (3)$$

where  $\bar{U}_{eff}(\varphi)$  is given by:

$$\bar{U}_{eff}(\varphi) = \frac{\langle f_{01} f_{02} | U_{ee}(|\vec{r}_{12}|) | f_{01} f_{02} \rangle}{\langle f_{01} f_{02} | f_{01} f_{02} \rangle} \quad (4)$$

Here  $f_{0i} = f_{0i}(z_i)$  is the one-electron ground state wave function for an electron in a one dimensional quantum well of wide equal to  $2h$  infinite height. By using the center-of-mass  $\theta = (\varphi_1 + \alpha^2 \varphi_2)/(\alpha^2 + 1)$  and relative  $\varphi = \varphi_1 - \varphi_2$  coordinates the two-particle Hamiltonian (3) can be separated into center-of-mass,  $\tilde{H}_c$  and relative  $\tilde{H}_\varphi$  terms:

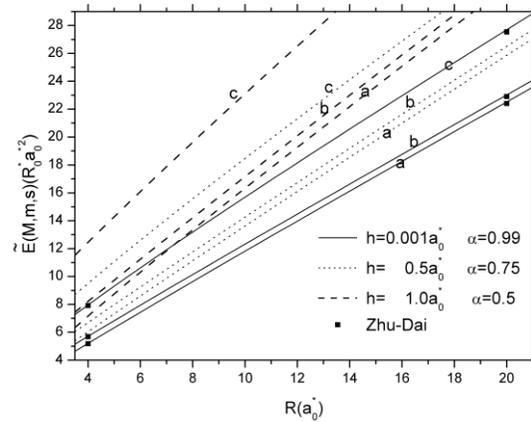
$$\tilde{H}_c = -\left( \frac{1}{\alpha^2 + 1} \right) \frac{\partial^2}{\partial \theta^2} - i\gamma R^2 \frac{\partial}{\partial \theta} + (\alpha^2 + 1) \frac{\gamma^2 R^4}{4} \quad (5)$$

$$\tilde{H}_\varphi = -\left( \frac{1}{\alpha^2 + 1} \right) \frac{\partial^2}{\partial \varphi^2} + \bar{U}_{eff}(\varphi) \quad (6)$$

This separability allows us to write the two-particle wave functions as  $\Psi_{M,m,s}(\theta, \varphi) = \exp(iM\theta)\Phi_{m,s}(\varphi)$  being  $M$  and  $m$  the center of mass and relative angular quantum numbers ( $M, m = 0, \pm 1, \pm 2, \dots$ ) and  $s = \pm$  the eigenfunction parity. The even solutions ( $s = +$ ) correspond to singlet states and the odd solutions ( $s = -$ ) correspond to triplet states. In our numerical work, we solve the wave equation  $\tilde{H}_\varphi \Phi_{m,s}(\varphi) = \tilde{E}(\varphi)(m, s)\Phi_{m,s}(\varphi)$  by using the trigonometric sweep method [11] with the periodical condition  $\psi_{m,s}(0) = (-1)^m \psi_{m,s}(2\pi)$ .

### 3. Results and Discussion

By comparing the results obtained in the present work with those previously reported in Ref [6, 10] is possible to establish the quality of the model and the numerical method used in this research. In this sense, we have systematically calculated the evolution of the normalized energies  $\tilde{E}(M, m, s) = \tilde{E}_c(M) + \tilde{E}_\varphi(m, s)$  of the first three low-lying states as a function of the inner radius  $R$  with some different pairs of values of the ratio between the radii and height of ribbons. The chosen values for  $\alpha$  are close to one ( $\alpha = 0.5, 0.75, 0.99$ ), while those of  $h$  are close to zero ( $h = 1.0, 0.5, 0.01 a_0^*$ ). In Figure 2 we display the curves obtained with this set of values of  $\alpha$  and  $h$ .



**Fig. 2.** Normalized energies a(0,0,+), b( $\pm$ 1,1,1) c(0,2,1) of electron-electron interaction as a function of the ribbon's inner radius. The solid square symbol corresponds to the values for exact solution. The lines for different values of  $\alpha(0.5, 0.75, 0.99)$  and  $h(1.0, 0.5, 0.01) a_0^*$  are our results.

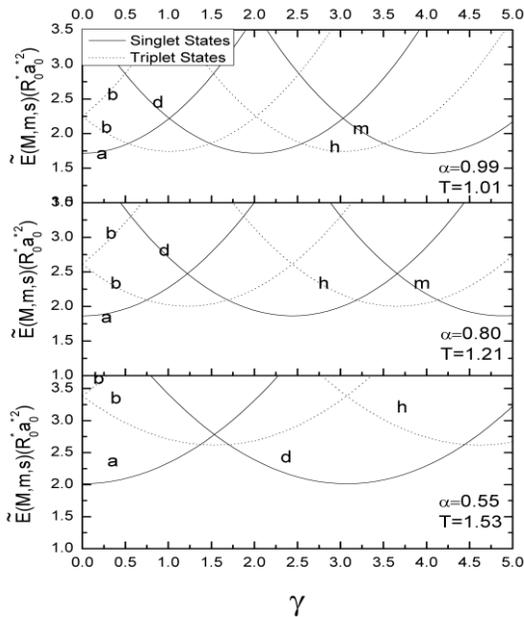
A glance on the curves of figure 2, for electrons spatially separated in two coaxial nanoribbons with inner radius  $\alpha R$  and outer radius  $R$ , allows us to conclude that the greater the height of the ribbons and the smaller the ratios between radius the higher are the corresponding energy levels. It is due to the fact that the largest possible separation between two electrons in two coaxial nanoribbons occurs for very large nanoribbons with small ratio between radii. When ratio between ribbon radii is always equal to one ( $\alpha = 0.99$ ), and the ribbon's height is always equal to zero ( $h = 0.01 a_0^*$ ), and the ribbons merge forming only one ring. This problem was exactly solved in Ref. [6], and their results for two different values of ring radius are shown in figure 2 by means of black square symbols. It can be seen that these energy values coincide with our results corresponding to the solid lines in figure 2, which were obtained by using a numerical procedure. This remarkable fact is a direct confirmation of the high quality of our results which was reconfirmed with the calculation of a broader set of energy levels as it is shown in Table 1. In the analysis of the low-lying energy states, we have used the same notation reported in reference [6]. We can observe that the results

display an asymptotic behavior when  $\alpha$  tends to one, and, simultaneously,  $h$  tends to zero. Additionally, it can be noticed that the two electron energy spectrum is strongly dependent on the geometrical parameters. For instance, all energy levels with inner radius equals to  $4a_0^*$  are smaller than those with inner radius equals to  $20a_0^*$ .

The same results occur when ratios between radii of the ribbons increase, and the ribbon's height decrease. This is because the renormalized potential energy is directly proportional to the inner radius, whereas is inversally proportional to the square root of  $h$ .

$\tilde{E}(M, m, s)$	One Dimensional Ring Ref. [6]		NANO-RIBBON					
	$4a_0^*$	$20a_0^*$	$\alpha = 0.99$ $h = 0.001a_0^*$		$\alpha = 0.75$ $h = 0.5a_0^*$		$\alpha = 0.5$ $h = 1.0a_0^*$	
			$4a_0^*$	$20a_0^*$	$4a_0^*$	$20a_0^*$	$4a_0^*$	$20a_0^*$
<b>a</b> (0,0,0)	5.18474	22.40328	5.214	22.522	6.044	25.868	7.172	30.827
<b>b</b> ( $\pm 1,1,1$ )	5.68474	22.90328	5.719	23.027	6.700	26.508	8.220	31.630
<b>c</b> (0,2,1)	7.91615	27.54199	7.916	27.702	9.520	32.331	12.459	39.768
<b>d</b> ( $\pm 2,0,0$ )	7.18474	24.40328	7.234	24.543	8.604	28.428	10.372	34.027
<b>e</b> ( $\pm 1,1,0$ )	8.41615	28.04199	8.469	28.207	10.041	32.971	11.712	40.521
<b>f</b> ( $\pm 2,2,1$ )	9.91615	29.54199	9.989	29.722	12.080	34.891	15.659	42.968
<b>g</b> (0,2,0)	11.40424	33.36149	11.469	33.569	13.499	39.690	15.014	49.593
<b>h</b> ( $\pm 3,1,1$ )	9.68474	26.90328	9.760	27.068	11.820	31.628	14.620	38.030
<b>i</b> ( $\pm 1,3,1$ )	11.90424	33.86149	11.992	34.074	14.621	40.331	20.132	50.701
<b>j</b> ( $\pm 3,1,0$ )	12.41615	32.04199	12.509	32.247	15.161	38.091	18.112	46.921
<b>k</b> ( $\pm 2,2,0$ )	13.40424	35.36149	13.489	35.589	16.059	42.250	18.214	52.793
<b>l</b> (0,4,1)	15.70729	39.89374	15.830	40.156	19.529	47.976	28.348	61.262
<b>m</b> ( $\pm 4,0,0$ )	13.18474	30.40328	13.294	30.603	16.284	36.108	19.972	43.627
<b>n</b> ( $\pm 1,3,0$ )	16.20729	40.39374	16.283	40.661	18.818	48.610	21.457	60.727
<b>o</b> ( $\pm 3,3,1$ )	15.90424	37.86149	16.033	38.114	19.741	45.451	26.532	57.101
<b>p</b> ( $\pm 4,2,1$ )	15.91615	35.54199	16.049	35.782	19.760	42.571	25.259	52.568
<b>q</b> ( $\pm 2,4,1$ )	17.70729	41.89374	17.850	42.177	22.089	50.536	31.548	64.462

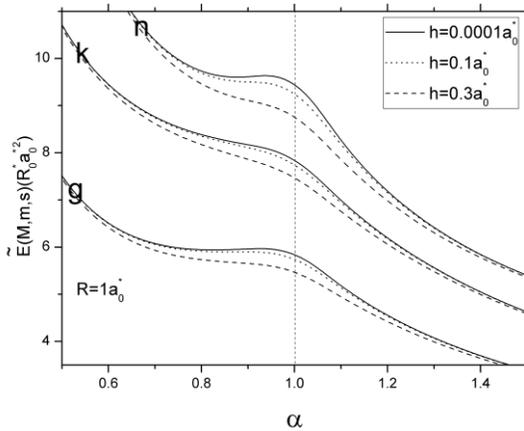
**Table 1.** Comparison between exact energy values corresponding to Ref. [6] and the energy values obtained for two electrons in two concentric quantum ribbons for a set of three different values of  $\alpha$  and  $h$ .



**Fig.3.** Several two-electron energy levels vs magnetic field.

In figure 3 we display a part of the two-electron energy spectrum as a function of the magnetic field which is threading the ribbons along the common symmetry axis. We have considered a set of three different values of the ratios between radii of the ribbons,  $\alpha = 0.99$  (upper panel),  $\alpha = 0.80$  (central panel), and  $\alpha = 0.55$  (lower panel). Only the states of  $|M| \leq 4$  are shown. The behaviour of these curves is the result of the strong competition between the paramagnetic and diamagnetic terms in the Hamiltonian (5). The contribution of diamagnetic term is quadratic with the magnetic field strength while the paramagnetic field is linear. For this reason, as the magnetic field strength is small, the curves have a constant slope, being positive for  $M > 0$ , and negative for  $M < 0$ ; they take a parabolic shape for large values of the magnetic field strength. These curves are different of those reported in a one dimensional ring [6], because the well-known Aharonov-Bohm oscillations corre-

sponding to periodic alternation of the ground state depends on both the inner radius, and the ratio between the radii and the oscillation. It is possible to observe that the greater the ratio between the radii of ribbons the smaller is the oscillation period regardless of the quantum ribbon's height. It is clear from the Eq. (5), since the oscillation period is inversally proportional to square of ratio between the radii. The analytic expression of the period can be written as  $T = 2M/R^2(1 + \alpha^2)$ . From these curves is possible to see that when the  $\alpha$  parameter begins to grow from  $\alpha = 0.55$  (lower panel) up to  $\alpha = 0.99$ , the triplet states begin to move toward higher values of the energy as a consequence of a further increase in the renormalizer potential energy which is in a good agreement with the Pauli's exclusion. Therefore, these results allows us to infer the existence of a transition between states strongly correlated, corresponding to very small or great values of  $\alpha$ , and weakly correlated states corresponding to values of  $\alpha$  close to one. In figure 4, we show total normalized energy of two-electrons for three different low-lying energy states as a function of ratio between the radii of the ribbons.

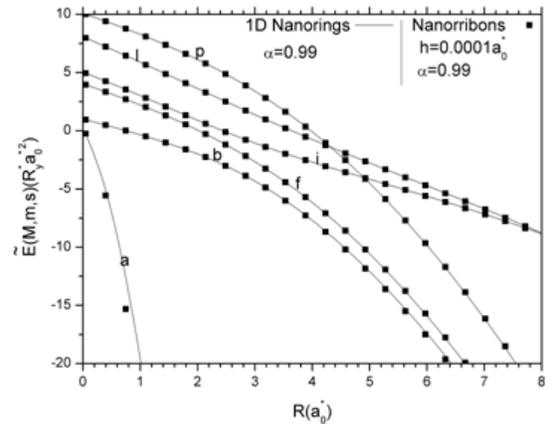


**Fig. 4.** Total renormalized energy of two-electrons in concentric quantum ribbons as a function of the ratio between the radii of the ribbons for three different values of the height:  $0.0001$ ,  $0.1$  and  $0.3a_0^*$

In order to give an explanation for the remarkable evolution of the energy dependencies on the ratio between radii of the ribbons, we should take into account that the renormalized potential energy is inversely proportional to  $\alpha$ , while the renormalized kinetic is independent of  $\alpha$ . For this reason, as  $\alpha$  is small the curves tend asymptotically to zero, but when  $\alpha$  takes values greater than one, the energy tends slowly to one. In these limit cases, the contribution of the renormalized potential energy to the total energy becomes very significant, while the renormalized energy is only important when the ribbons merge to one, which occurs as  $\alpha$  is close to one due to the electrons are constrained to be closer to each other. The asymptotic behavior of the total renormalized energy for small, and great values of  $\alpha$  is an

indicative of ordered system with particles strongly correlated similar to the crystal Wigner. From the curves displayed in figure 4, we can see that ribbon's height also affect the energy spectrum of the two electrons. Indeed, this effect is not large enough owing to the radius of the ribbon is much greater than the height. Nevertheless, we could expect that this effect changes as the height becomes greater than the radius.

Finally, the model studied in this work is very flexible, and it allow us to consider attractive energy potential by taking a particles with charges  $e$  and  $-e$  similar to excitons. Instead of the repulsive energy potential given by the Eq. (2), we consider the attractive Coulomb interaction which becomes equal to  $-\tilde{U}(|\vec{r}_{12}|)$ . In order to simplify the analysis, we only take into account two particles with the same effective masses. In figure 5, we present a part of the electron-hole energy spectrum as a function of the inner radius for a very small height,  $\alpha = 0.99$ , and zero magnetic field strength. These values were considered in order to compare our results with those previously reported in Ref. [10]. Our results correspond to the black square symbols, and the solid curves are the results of Ref. [10].



**Fig. 5.** Total renormalized energy of an electron-hole as function of the inner radius.

We can observe the total coincidence between the two sets of results, which show the high degree of flexibility in our model. When the inner radius is small the slope of curves change, but this slope becomes a constant when the inner radius is large enough. This situation describes a transition between a disordered system, similar to a gas ( $\tilde{E}_K > \tilde{U}$ ), and an ordered system similar to an ionic crystal ( $\tilde{U} > \tilde{E}_K$ ).

## Conclusions

In short, we propose a simple method for analyzing the energy spectrum of two particles spatially separated, and constrained to move in ribbons with cylindrical symmetry. We show that our results, in limit cases, are in a good

agreement with those previously reported. Preliminary numerical results show that any height of the ribbons in the Aharonov–Bohm oscillations are strongly dependent on the ratio between radii of the rings of the lower energy levels of the exciton.

### Acknowledgements

The authors wish to thank the Universidad Nacional for the economical aids. This work was partially supported by Universidad Nacional de Colombia sede Medellín through the DIME Cod. QUIPU 20101007748.

### References

- [1] M. Susuki, D. Tsuya, S. Moriyoma, T. Fuse, and K. Shisbashi, *Physica E* **24**, 2004, p. 10.
- [2] V. Ya Prinz, *Physica E*, **23**, 2004, p. 260.
- [3] V. Ya Prinz, *Physica E* **24**, 2004, p. 54.
- [4] A. B. Vorob'ev, V. Ya Prinz, Yu. S. Yukecheva, A. I. Toropov, *Physica E*, **23**, 2004, p. 171.
- [5] G. Ferrari, A. Bertoni, G. Goldoni, and E. Molinari, *Phys. Rev. B*, **78**, 2008, 115326.
- [6] J.-L. Zhu, Z. Dai, X. Hu, *Phys. Rev. B* **68** (2003) 045324.
- [7] J. H. Marín, L. F. García, and I. D. Mikhailov, *Braz. Jour. Of Phys.* **36**, 2006 3B.
- [8] L. Jacak, P. Hawrylak, and A. Wojs, *Quantum Dots*, Berlin, Springer, 1997.
- [9] T. Mano, T. Kuroda, S. Sanguinetti, T. Ochiai, T. Tateno, J. Kim, T. Noda, M. Kawabe, K. Sakoda, G. Kido, and N. Koguchi, *Nano Lett.* **5**, 2005, p. 425.
- [10] L. F. García, I. D. Mikhailov, and J. H. Marín, *Rev. Col. de Fís.* **38**, 2006, p. 1078.
- [11] F. M. Peeters and V. A. Schweigert, *Phys. Rev. B*, **53**, 1996, p. 1468; I. D. Mikhailov, J. H. Marín, and F. García, *Phys. Stat. Sol. (b)*, **242**, 2005, p. 1636.